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Perturbations of Hemivariational Inequalities with Constraints and Applications

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Abstract. The aim of the present paper is to discuss the influence which certain perturbations have on the solution of the eigenvalue problem for hemivariational inequalities on a sphere of given radius. The perturbation results in adding a term of the type $g^0(x, u(x); v(x))$ to the hemivariational inequality, where g is a locally Lipschitz nonsmooth and nonconvex energy functional. Applications illustrate the theory.

Key words: Hemivariational inequalities, Perturbation, Deformation lemma, Critical points.

Introduction

The study of variational inequalities began in the early sixties with the pioneering works of G. Fichera [8], J.L. Lions and G. Stampacchia [11]. The connection of this theory with the notion of the subdifferential of a convex function was achieved by J.J. Moreau [12], who introduced the notion of convex superpotentials.

The mathematical theory of hemivariational inequalities, as well as their applications in Mechanics, Engineering or Economics, were introduced and developed by P. D. Panagiotopoulos [20–27] in the case of nonconvex energy functions. He also defined the notion of nonconvex superpotentials [19]. An overview of these methods is given in the recent monograph by Z. Naniewicz and P. D. Panagiotopoulos [16]. By replacing the subdifferential of a convex function by the generalized gradient (in the sense of F.H. Clarke) of a locally Lipschitz functional, hemivariational inequalities arise whenever the energetic functional associated to a concrete problem is nonconvex. The hemivariational inequalities appear as a generalization of the variational inequalities, but they are much more general than these ones, in the sense that they are not equivalent to minimum problems but, they give rise to substationarity problems. Since one of the main ingredients of this study is based on the notion of Clarke subdifferential of a locally Lipschitz functional, the theory of hemivariational inequalities appears as a new field of Non-smooth Analysis.

Note that all problems formulated in terms of hemivariational inequalities can be formulated "equivalently" as multivalued differential equations. However, the formulation in terms of hemivariational inequalities has a great advantage: that the hemivariational inequalities express a physical principle, the principle of virtual work or power. This fact permits us to use all the advantages of the energetic approach in the mathematical treatment. Moreover, the energetic approach is the only approach towards the development of a solid numerical method.

1. The abstract framework

Let V be a real Hilbert space with the scalar product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|$. Assume V is densely and compactly imbedded in $L^p(\Omega;\mathbf{R}^N)$, for some $1 < p < +\infty$ and $N \geq 1$, where Ω is a bounded domain in $\mathbf{R}^m$, $m \geq 1$. In particular, the continuity of this embedding ensures the existence of a positive constant $C_p(\Omega)$ such that

$$\|u\|_{L^p} \leq C_p(\Omega)\|u\|, \quad \text{for all } u \in V.$$

Throughout, the Euclidean norm in $\mathbf{R}^N$ will be denoted by $|\cdot|$, while the duality pairing between V^* and V (resp., between $(\mathbf{R}^N)^*$ and $(\mathbf{R}^N)$ will be denoted by $\langle\cdot,\cdot\rangle_V$ (resp., $\langle\cdot,\cdot\rangle$).

Let $a : V \times V \to \mathbf{R}$ be a continuous, symmetric and bilinear form, which is not necessarily coercive. Let $A : V \to V^*$ be the self-adjoint bounded linear operator which corresponds to a, that is, for every $u, v \in V$,

$$\langle Au, v\rangle_V = a(u,v).$$

For $r > 0$, set S_r the sphere of radius r in V centered at the origin, i.e.

$$S_r = \{u \in V; \|u\| = r\}.$$

Consider a mapping $C : S_r \times V \to \mathbf{R}$, to which we impose no continuity assumption. However, for our purpose, a weak kind of compactness hypothesis is given by

(H$_1$) There exists a locally Lipschitz function $f : V \to \mathbf{R}$, even and bounded on S_r, satisfying

$$C(u,v) \geq f^0(u;v), \quad \text{for all } (u,v) \in S_r \times V, \text{ with } (u,v) = 0,$$

and such that the set

$$\{\zeta \in V^*; \zeta \in \partial f(u), u \in S_r\}$$

is relatively compact in V^*.

Here $f^0(u;v)$ stands for the Clarke derivative of f at $u \in V$ with respect to the direction $u \in V, v \neq 0$, that is

$$f^0(u;v) = \limsup_{\substack{w\to u\\ \lambda\searrow 0}} \frac{f(w+\lambda v) - f(w)}{\lambda}.$$

Accordingly, Clarke's generalized gradient $\partial f(u)$ of f at u is defined by

$$\partial f(u) = \{\zeta \in V^*; f^0(u;v) \geq \langle \zeta, v\rangle_V, \quad \text{for all } v \in V\}.$$

Let $j : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ be a Carathéodory function which is locally Lipschitz with respect to the second variable and such that $j(\cdot, 0) \in L^1(\Omega)$. We also assume that this functional satisfies the symmetry condition
(H$_2$) $j(x, y) = j(x, -y)$, for a.e. $x \in \Omega$ and every $y \in \mathbf{R}^N$.
and
(H$_3$) there exist $a_i \in L^{p/(p-1)}(\Omega)$ and $b \in \mathbf{R}$ such that

$$|w| \leq a_1(x) + b|y|^{p-1}, \quad \text{for a.e. } (x, y) \in \Omega \times \mathbf{R}^N \text{ and all } w \in \partial j(x, y).$$

We have denoted by $\partial j(x, y)$ Clarke's generalized gradient of the locally Lipschitz mapping $y \mapsto j(x, y)$, for some fixed $x \in \Omega$.

Let $\Lambda : V \to V^*$ be the duality isomorphism

$$(\Lambda u, u)_V = (u, v), \quad \text{for all } u, v \in V.$$

Our last assumption is **(H$_4$)** Let $(u_n) \subset S_r$ be an arbitrary sequence which converges weakly in V to some u. Consider a sequence $\zeta_n \in \partial f(u_n)$ such that

$$a(u_n, u_n) + \langle \zeta_n, u_n\rangle_V \to \alpha_0$$

and, for every $w \in L^{p/(p-1)}(\Omega; \mathbf{R}^N)$ verifying

$$w(x) \in \partial j(x, u(x)), \quad \text{for a.e. } x \in \Omega,$$

the sequence $\{(A - \lambda_0 \Lambda)u_n\}$ is convergent. Then there exists a strongly convergent subsequence of (u_n) in V. Here λ_0 is defined by

$$\lambda_0 = r^{-2}\left(\alpha_0 + \int_\Omega \langle w(x), u(x)\rangle \, \mathrm{d}x\right).$$

In the proof of our main result we shall make use of some notions of Algebraic Topology, for which we refer to [29, Chapter 1] (see also [6, 7]). We recall only few basic definitions.

Let X be a metric space and $A \subset X$. A map $r : X \to A$ is said to be a *retraction* if it is continuous, surjective and $r_{|A} = Id$. A retraction r is called to be a *strong deformation retraction* provided there exists a homotopy $F : X \times [0, 1] \to X$ of $i \circ r$ and Id_X which satisfies the additional condition $F(x, t) = F(x, 0)$, for each $(x, t) \in A \times [0, 1]$. Here i stands for the inclusion map of A in X. The metric space X is said to be *weakly locally contractible*, if every point has a neighbourhood which is contractible in X.

Let $\psi : X \to \mathbf{R}$ be a locally Lipschitz functional. For every $a \in \mathbf{R}$, set

$$[\psi \leq a] = \{u \in X; \psi(u) \leq a\}.$$

Given $a, b \in \mathbf{R}$ with $a \le b$, the pair $([\psi \le b], [\psi \le a])$ is said to be *trivial* provided that, for every neighbourhood $[a', a'']$ of b, there exist some closed sets A and B such that $[\psi \le a'] \subset A \subset [\psi \le a'']$, $[\psi \le b'] \subset B \subset [\psi \le b'']$ and such that A is a strong deformation retract of B.

A real number c is said to be an *essential value* of ψ if, for every $\varepsilon > 0$, there exist $a, b \in (c - \varepsilon, c + \varepsilon)$, with $a < b$ and such that the pair $([\psi \le b], [\psi \le a])$ is not trivial. This notion is essentially due to M. Degiovanni and S. Lancelotti [7].

2. The main result

Let us consider the following eigenvalue hemivariational inequality with constraints:

(P$_1$) Find $(u, \lambda) \in V \times \mathbf{R}$ such that, for all $v \in V$,

$$\begin{cases} a(u,v) + C(u,v) + \int_\Omega j^0(x, u(x); v(x))\, \mathrm{d}x \ge \lambda(u,v), \\ \|u\| = r. \end{cases} \tag{1}$$

Under hypotheses **(H$_1$)**–**(H$_4$)**, Motreanu and Panagiotopoulos proved in [13, Theorem 4] that this problem admits infinitely many pairs of solutions $(\pm u_n, \lambda_n)$, with all u_n distinct. Moreover, they find the expression of eigenvalues λ_n. Remark that their statement is done under a slight less general hypothesis, namely by assuming $a_i =$ const. in **(H$_3$)**. Examining this proof, we remark that in order to show that the arguments of [13] hold in our case, it is sufficient to verify that the energy functional

$$F(u) = \tfrac{1}{2}\, a(u,u) + f(u) + J(u), \quad u \in V, \tag{1a}$$

is bounded from below on S_r where $J : L^p(\Omega; \mathbf{R}^N) \to \mathbf{R}$ is defined by $J(u) = \int_\Omega j(x, u(x))\, \mathrm{d}x$. Indeed, notice first that, for a.e. $(x, y) \in \Omega \times \mathbf{R}^N$,

$$\begin{aligned} |j(x,y)| &\le |j(x,0)| + |j(x,y) - j(x,0)| \\ &\le |j(x,0)| + \sup\{|w|; w \in \partial j(x,Y),\ Y \in [0,y]\} \cdot |y| \\ &\le |j(x,0)| + \alpha_1(x)|y| + b|y|^p\,. \end{aligned}$$

Therefore

$$|J(u)| \le \|j(\cdot,0)\|_{L^1} + \|a_1\|_{L^{p'}} \cdot \|u\|_{L^p} + b\|u\|^p_{L^p}.$$

Hence,

$$\begin{aligned} F_{|S_r}(u) \ \ge\ & -\tfrac{1}{2}\|a\| \cdot r^2 - \|f\|_{L^\infty} - \|j(\cdot,0)\|_{L^1} \\ & -C_p(\Omega)\|a_1\|_{L^{p'}} r - bC_p^p(\Omega) r^p. \end{aligned}$$

From now on the proof follows from the same lines as in [13].

A natural question arises now: what happens if we perturb (1) in a suitable manner? Perturbation results for the case of equations have been established in [1, 2] while perturbation techniques for variational inequalities have been developed in [3, 7]. Let us consider the following non-symmetric perturbed hemivariational inequality:

(P$_2$) Find $(u, \lambda) \in V \times \mathbf{R}$ such that

$$\begin{cases} a(u,v) + C(u,v) + \int_\Omega (j^0(x,u(x);v(x)) + g^0(x,u(x);v(x)))\,\mathrm{d}x \\ +\langle \varphi, v\rangle_V \geq \lambda(u,v), \quad \text{for all } u \in V \\ \|u\| = r, \end{cases} \tag{2}$$

where $\varphi \in V^*$ and $g : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function which is locally Lipschitz with respect to the second variable and such that $g(\cdot, 0) \in L^1(\Omega)$. Fix $\delta > 0$. We make no symmetry assumption on g, but we impose only the growth condition **(H$_5$)** $|w| \leq a_2(x) + \delta|y|^{p-1}$, for a.e. $(x, y) \in \Omega \times \mathbf{R}^N$ and for all $w \in \partial g(x, y)$, where $a_2 \in L^{p/(p-1)}(\Omega)$.

We also assume

(H$_6$) The mappings $g(\cdot, 0)$, a_2 and φ satisfy

$$\|a_2\|_{L^{p'}} \leq \delta \quad \text{and} \quad \|\varphi\|_{V^*} \leq \delta.$$

As a compactness condition we assume the following variant of **(H$_4$)**:

(H$_7$) Let $(u_n) \subset S_r$ be an arbitrary sequence which converges weakly in V to some u. Assume $\zeta_n \in \partial f(u_n)$ such that

$$a(u_n, u_n) + \langle \zeta_n, u_n\rangle_V \to \alpha_0$$

and, for every $w, z \in L^{p/(p-1)}(\Omega; \mathbf{R}^N)$ verifying

$$w(x) \in \partial j(x, u(x)) \quad \text{and} \quad z(x) \in \partial g(x, u(x)) \quad \text{for a.e. } x \in \Omega, \tag{2a}$$

the sequence $\{(A - \lambda_0\Lambda)u_n\}$ is convergent. then (u_n) is relatively compact in V. Here λ_0 is defined by

$$\lambda_0 = r^{-2}\left(\alpha_0 + \int_\Omega \langle w(x) + z(x), u(x)\rangle\,\mathrm{d}x\right).$$

Our aim is to show that the number of solutions of **(P$_2$)** increases as $\delta \to 0$. More precisely, we have

THEOREM 1. *Assume hypotheses* (**H$_1$**)–(**H$_7$**) *hold. Then, for every* $n \geq 1$, *there exists* $\delta_n > 0$ *such that, for each* $\delta \leq \delta_n$, *the problem* (**P$_2$**) *admits at least* n *distinct solutions.*

In the proof of our main result, given in the next section, we shall make use of some techniques from [6, 7, 13, 15].

3. Proof of Theorem 1

We shall follow in the proof a method developed by Degiovanni and Lancelotti in [7].

For every $n \geq 1$, set

$$\Gamma_n = \{S \subset S_r; S \in \mathcal{F}, \gamma(S) \geq n\},$$

where $\mathcal{F}$ denotes the family of closed and symmetric subsets of S_r with respect to the origin and $\gamma(S)$ represents Krasnoselski's genus of the set $S \in \Gamma_n$. Namely, $\gamma(S)$ is the smallest $k \in \mathbf{N} \cup \{+\infty\}$ for which there exists an odd continuous mapping from S into $\mathbf{R}^k \setminus \{0\}$. Motreanu and Panagiotopoulos proved in [13] that the corresponding min-max values of F over Γ_n

$$\beta_n = \inf_{S \in \Gamma_n} \sup_{u \in S} F(u), \quad n \geq 1,$$

are critical values of F on S_r. We first remark that

LEMMA 1. *We have that* $\sup_{S_r} F$ *is not achieved and* $\lim_{n\to\infty} \beta_n = \sup_{u \in S_r} F(u)$. *Moreover, there exists a sequence* (b_n) *of essential values of* $F_{|S_r}$ *strictly increasing to* $\sup_{u \in S_r} F(u)$.

Proof. The proof of this result is essentially contained in [7]. It is sufficient to adapt the arguments given in these papers for the case of locally Lipschitz functionals and replacing the classical Fréchet-differentiability by the subdifferentiability in the sense of Clarke. We point out only the main steps of the proof:

(i) The functional $F_{|S_r}$ satisfies the Palais–Smale condition (see the proof of Theorem 4 in [13]). So, if there exist $u_0 \in S_r$ and $m < n$ such that $\beta_m = \beta_n \leq f(u_0)$, then $\gamma(K_{\beta_m}) \geq n - m + 1$, where

$$K_{\beta_m} = \{u \in S_r; F(u) = \beta_m \quad \text{and} \quad \lambda_F(u) = 0\}.$$

In the above relation, λ_F is defined by

$$\lambda_F(u) = \min\{\|\xi\|; \xi \in \partial F(u)\}.$$

It is known (see [4]) that if F is a locally Lipschitz functional then λ_F is lower semi-continuous.

(ii) If the sequence (β_n) is stationary and if there exists $u_0 \in S_r$ such that (i) holds, then $\gamma(K_{\beta_m}) = +\infty$, for some $m \geq 1$. This is not possible, since S_r is a weakly locally contractible space and K_{β_m} is a compact set, which implies $\gamma(K_{\beta_m}) < +\infty$.

(iii) It follows by the previous steps, the definition of Krasnoselski's genus and the fact that $F \not\equiv$ const. on S_r, that $\sup_{u \in S_r} F(u)$ is not achieved and $\lim_{n\to\infty} \beta_n = \sup_{u \in S_r} F(u)$. Moreover, without loss of generality, we may assume that $\sup_{u \in S_r} F(u) = +\infty$. Let us define

$$\bar{\Gamma}_n = \{\varphi(S^{n-1}); \varphi : S^{n-1} \to S_r \text{ is continuous and odd}\},$$

and

$$\bar{\beta}_n = \inf_{C \in \bar{\Gamma}_n} \sup_{u \in C} F(u).$$

Of course, $\bar{\beta}_n \geq \beta_n$, so that $\lim_{n\to\infty} \bar{\beta}_n = \sup_{u \in S_r} F(u) = +\infty$. By Theorem 2.12 of [7] it follows that there exists a sequence $9_n)$ of essential values of $F_{|S_r}$ strictly increasing to $\sup_{u \in S_r} F(u)$. □

Notice that the proof of Theorem 4 in [13] works if f is supposed to be only bounded from below on S_r. If sup $f(u)_{u \in S_r} = \infty$ then $\sup_{u \in S_r} F(u) = \infty$ and $\beta_n \to \infty$, as $n \to \infty$.

We associate to the hemivariational problem (**P**$_2$) the energy function $H : V \to$ **R**, defined by

$$H(u) = \frac{1}{2} a(u, u) + f(u) + J(u) + G(u) + \langle \varphi, u \rangle_V, \tag{2b}$$

where $G(u) = \int_\Omega g(x, u(x))\, \mathrm{d}x$, for every $u \in L^p(\Omega; \mathbf{R}^n)$. The next result asserts that if δ is chosen sufficiently close to 0 in (**H**$_5$) and (**H**$_6$), then H is a small perturbation of the functional F on S_r.

LEMMA 2. *For every* $\varepsilon > 0$, *there exists* $\delta_0 > 0$ *such that, for all* $\delta \leq \delta_0$,

$$\sup_{u \in S_r} |F(u) - H(u)| < \varepsilon.$$

Proof. We have

$$|g(x, y)| \leq |g(x, 0)| + a_2(x)|y| + \delta |y|^p.$$

Thus, for all $u \in S_r$,

$$\begin{aligned} |F(u) - H(u)| &\leq |G(u) + \langle \varphi, u \rangle_V| \leq |G(u)| + \delta r \\ &\leq \|g(\cdot, 0)\|_{L^1} + \delta C_p(\Omega) r + \delta C_p^p(\Omega) r^p + \delta r < \varepsilon, \end{aligned}$$

for $\delta > 0$ small enough. □

LEMMA 3. *The functional* H *satisfies the Palais–Smale condition on* S_r.

Proof. Let (u_n) be a sequence in S_r such that $\sup_n |H(u_n)| < +\infty$ and $\lambda_H(u_n) \to 0$, as $n \to \infty$. The expression of the generalized gradient of H on S_r is given by

$$\partial(H_{|S_r}(u) = \{\xi - r^{-2} \langle \xi, u \rangle_V \lambda u; \xi \in \partial H(u)\}. \tag{3}$$

Consequently, there exists a sequence $(\xi_n) \subset V^*$ such that

$$\xi_n \in \partial H(u_n) \tag{4}$$

and

$$\xi_n - r^{-2}\langle \xi_n, u_n\rangle_V \Lambda u_n \to 0, \quad \text{strongly in } V^*. \tag{5}$$

We have to prove that (u_n) is relatively compact.

Using (4), (5) and applying the formula for the generalized gradient of a sum (see, e.g., [5, Proposition 2.3.3]) in the expression of H, one obtains the existence of $\zeta_n \in \partial f(u_n), w_n \in \partial(J_{|V})(u_n)$ and $z_n \in \partial(G_{|V})(u_n)$ such that

$$Au_n + \zeta_n + w_n + z_n - r^{-2}\langle u_n + \zeta_n + w_n + z_n, u_n\rangle_V \Lambda u_n + \varphi \to 0 \quad \text{strongly in } V^*. \tag{6}$$

Moreover, the density of V in $L^p(\Omega; \mathbf{R}^N)$ implies (see [4, Theorem 2.2])

$$\partial(J_{|V})(u) \subset \partial J(u) \quad \text{and} \quad \partial(G_{|V})(u) \subset \partial G(u).$$

It is well known that the embedding $V^* \subset L^{p/(p-1)}(\Omega; \mathbf{R}^N)$ is compact. Thus one can suppose, passing eventually to subsequences, that

$$w_n \to w \quad \text{strongly in } V^* \tag{7}$$

$$z_n \to z \quad \text{strongly in } V^*. \tag{8}$$

Furthermore, hypothesis ($\mathbf{H}_4$) implies that (eventually, at a subsequence),

$$\zeta_n \to \zeta \quad \text{strongly in } V^*. \tag{9}$$

Since $\|u_n\| = r$, we can also assume that

$$u_n \to u \quad \text{weakly in } V. \tag{10}$$

Additionally, we can suppose that

$$\{a(u_n, u_n)\} \quad \text{converges in } \mathbf{R} \tag{11}$$

and

$$\langle w_n + z_n, u_n\rangle_V \to \langle w + z, u\rangle_V. \tag{12}$$

Using the upper semicontinuity of the Clarke generalized gradient (see [5, Proposition 2.1.5]), the relations (6), (12) and the hypothesis ($\mathbf{H}_1$), we find

$$w \in \partial(J_{|V})(u) \tag{13}$$

$$z \in \partial(G_{|V})(u) \tag{14}$$

$$\zeta \in \partial f(u). \tag{15}$$

Applying now Theorem 2.7.5 in [5], the relations (13) and (14) yield

$$w(x) \in \partial j(x, u(x)) \quad \text{for a.e.} x \in \Omega \tag{16}$$

$$z(x) \in \partial g(x, u(x)) \quad \text{for a.e.} x \in \Omega. \tag{17}$$

Set

$$\lambda_0 = r^{-2} \left(\alpha_0 + \int_\Omega \langle w(x) + z(x), u(x) \rangle \, \mathrm{d}x \right),$$

where

$$\alpha_0 = \lim_{n\to\infty} \{a(u_n, u_n) + \langle w_n + z_n, u_n \rangle_V\}.$$

Relations (6)–(12) allow us now to deduce that the sequence $\{(A - \lambda_0 \Lambda)u_n\}$ converges strongly in v^*. Then, by (**H**$_7$), there exists a strongly convergent subsequence of (u_n), which concludes our proof. □

LEMMA 4. *If u is a critical point of $H_{|S_r}$ then there exists $\lambda \in \mathbf{R}$ such that (u, λ) is a solution of* $(\mathbf{P}_2)$.

Proof. We have, for every $u \in V$,

$$\partial H(u) = \lambda Au + \partial(J_{|V})(u) + \partial(G_{|V})(u) + \varphi. \tag{18}$$

Since $0 \in \partial(H_{|S_r})(u)$, it follows by (3) and (14) that there exists

$$w \in \partial(J_{|V})(u) \subset \partial J(u) \quad \text{and} \quad z \in \partial(G_{|V})(u) \subset \partial G(u) \tag{19}$$

such that u is a solution of

$$\Lambda Au + w + z + \varphi = r^2 \langle Au + w + z + \varphi, u \rangle_V. \tag{20}$$

Moreover (see [5, Theorem 2.7.3]), for every $u \in L^p(\Omega; \mathbf{R}^N)$,

$$\partial J(u) \subset \int_\Omega \partial j(x, u(x)) \, \mathrm{d}x \quad \text{and} \quad \partial G(u) \subset \int_\Omega \partial g(x, u(x)) \, \mathrm{d}x.$$

Thus, by (19), the mappings $w, z : \Omega \to (\mathbf{R}^N)^*$ satisfy

$$w(x) \in \partial j(x, u(x)) \quad \text{for a.e. } x \in \Omega, \tag{21}$$

$$z(x) \in \partial g(x, u(x)) \quad \text{for a.e. } x \in \Omega, \tag{22}$$

and, for all $v \in V$,

$$\langle w, v\rangle_V = \int_\Omega \langle w(x), v(x)\rangle \,\mathrm{d}x, \tag{23}$$

$$\langle z, v\rangle_V = \int_\Omega \langle z(x), v(x)\rangle \,\mathrm{d}x. \tag{24}$$

Set

$$\lambda = r^{-2}(\langle \Lambda A u + \varphi, u\rangle_V = \int_\Omega \langle w(x) + z(x), u(x)\rangle \,\mathrm{d}x. \tag{25}$$

It follows by (20)–(25) that, for every $v \in V$,

$$\begin{aligned} \lambda(u,v) - a(u,v) - \langle\varphi, u\rangle_V &= \int_\Omega \langle w(x) + z(x), v(x)\rangle \,\mathrm{d}x \\ &\le \int_\Omega \max\{\langle \mu, v(x)\rangle; \mu \in \partial(w+z)(x,u(x))\}\,\mathrm{d}x \\ &\le \int_\Omega \max\{\langle \mu_1, v(x)\rangle; \mu_1 \in \partial w(x,u(x))\}\,\mathrm{d}x \\ &\quad + \int_\Omega \max\{\langle \mu_2, v(x)\rangle; \mu_2 \in \partial z(x,u(x))\}\,\mathrm{d}x \\ &= \int_\Omega j^0(x,u(x);v(x))\,\mathrm{d}x + \int_\Omega g^0(x,u(x);v(x))\,\mathrm{d}x. \end{aligned} \tag{26}$$

We have used above the classical inclusion (see [5, Proposition 2.3.3])

$$\partial(w+x)(x,u(x)) \subset \partial w(x,u(x)) + \partial z(x,u(x)).$$

We point out that the last equality in (26) holds because of Proposition 2.1.2 from [5]. □

PROOF OF THEOREM 1. Fix $n \ge 1$. Taking into account Lemma 4, it suffices to motivate the existence of some $\delta_n > 0$ such that, for every $\delta \le \delta_n$, the functional $H_{|S_r}$ has at least n distinct critical values.

By Lemma 1, let (b_n) be a sequence of essential values of $F_{|S_r}$ strictly increasing to $\sup_{u\in S_r} F(u)$. Fix $n \ge 1$ and some $\varepsilon_0 < 1/2 \min_{1\le j\le n-1}(b_{j+1} - b_j)$. We apply Theorem 2.6 from [7] to $F_{|S_r}$ and $H_{|S_r}$. Hence, for every $1 \le j \le n-1$, there exists $\eta_j > 0$ such that

$$\sup_{u\in S_r} |F(u) - H(u)| < \eta_j$$

implies the existence of an essential value c_j of $H_{|S_r}$ in $(b_j - \varepsilon_0, b_j + \varepsilon_0)$. We now apply Lemma 2 for $\varepsilon = \min\{\varepsilon_0, \eta_1, \ldots, \eta_{n-1}\}$. This yields the existence of some $\delta_n > 0$ such that

$$\sup_{u\in S_r} |F(u) - H(u)| < \varepsilon,$$

provided $\delta \le \delta_n$ in (**H**$_5$) and (**H**$_6$). So, we have obtained that the functional $H_{|S_r}$ has at least n distinct essential values $c_1, \ldots, c_n$ in $(-\infty, b_n + \varepsilon)$. It remains to prove that $c_1, \ldots, c_n$ are critical values of $H_{|S_r}$. Arguing by contradiction, let us assume that c_j is not a critical value of $H_{|S_r}$.

CLAIM 1. *There exists* $\varepsilon > 0$ *so that* $H_{|S_r}$ *has no critical value in* $(c_j - \varepsilon, c_j + \varepsilon)$.

PROOF OF CLAIM. Indeed, if not, there is a sequence (d_n) of critical values of $H_{|S_r}$ with $d_n \to c_j$, as $n \to \infty$. Since d_n is a critical value, there exists $u_n \in S_r$ such that

$$H(u_n) = d_n \quad \text{and} \quad \lambda_H(u_n) = 0.$$

Now we take into account that $(PS)_{c_j}$ holds. Therefore, up to a subsequence, one can suppose that (u_n) converges to some $u \in S_r$, as $n \to \infty$. By the continuity of H and the lower semi-continuity of λ_H, it follows that

$$H(u) = c_j \quad \text{and} \quad \lambda_H(u) = 0,$$

which contradicts the initial assumption on c_j and concludes the proof of our Claim.

Now we apply the Noncritical Point Theorem (see [6, Theorem 2.15]), which can be also deduced as a consequence of the Deformation Lemma for locally Lipschitz functionals (see [4, Theorem 3.1]). Thus, for some fixed $c_j - \varepsilon < a < b < c_j + \varepsilon$, there exists a continuous map $\eta : S_r \times [0,1] \to S_r$ such that, for each $(u,t) \in S_r \times [0,1]$,

$$\eta(u,0) = u, \quad H(\eta(u,t)) \le H(u),$$

$$H(u) \le b \Rightarrow H(\eta(u,1)) \le a, \quad H(u) \le a \Rightarrow \eta(u,t) = u.$$

It follows that the map

$$\rho : [H_{|S_r} \le b] \to [H_{|S_r} \le b], \quad \rho(u) = \eta(u,1)$$

is a retraction. Set

$$\mathcal{H} : [H_{|S_r} \le b] \times [0,1] \to [H_{|S_r} \le b], \quad \mathcal{H}(u,t) = \eta(u,t).$$

We observe that, for every $u \in [H_{|S_r} \le b]$,

$$\mathcal{H}(u,0) = u \quad \text{and} \quad \mathcal{H}(u,1) = \rho(u). \tag{27}$$

Moreover, for each $(u,t) \in [H_{|S_r} \le a] \times [0,1]$,

$$\mathcal{H}(u,t) = \mathcal{H}(u,0). \tag{28}$$

By (27) and (28) it follows that $\mathcal{H}$ is $[H_{|S_r} \le a]$-homotopic to the identity of $[H_{|S_r} \le a]$, i.e., $\mathcal{H}$ is a strong deformation retraction. This means that the pair

$([H_{|S_r} \leq b], [H_{|S_r} \leq a])$ is trivial. Therefore, c_j is not an essential value of $H_{|S_r}$. This contradiction concludes our proof. □

4. A note of the possible applications

The perturbation results obtained in the previous Sections may have a serious application in the study of the eigenvalue problems for hemivariational inequalities. Suppose, for instance, that we deal with the eigenvalue problem of two adhesively connected v. Kármán plates [28] and that the interface law has a very complicated form (a zig-zag nonmonotone multivalued diagram). Then one can consider the eigenvalue problem for a simplified interface law which results by "smoothing some parts" of the complicated initial law. With respect to the corresponding nonsmooth nonconvex potential energy (1a) this "simplification procedure" means that we have added an additional nonconvex and nonsmooth energy term (cf. Equation (2b)). The simplified interface law results by the "superposition" of the two nonmonotone multivalued relations given in (2a).

Here we deal with systems having a prescribed cost or weight or consumed energy. this is the meaning of the constraint $\|u\| = r$ and therefore we have an eigenvalue problem for hemivariational inequality on a sphere of a given radius.

Theorem 1 of the present paper holds in all cases of the applications given in [13] Section 3, where we refer the reader for further information.

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